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Chiral Symmetry Breaking by a Magnetic Field in Weak-coupling QED

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ABSTRACT

Using the nonperturbative Schwinger-Dyson equation, we show that chiral symmetry in weak-coupling massless QED is dynamically broken by a constant but arbitrarily strong external magnetic field.

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Chiral symmetry plays an important role in elementary particle and nuclear physics. In this Letter we examine its breaking in the theory of quantum electrodynamics. It has been known[1] for some time that QED may have a nonperturbative strong-coupling phase, characterized by spontaneous chiral symmetry breaking, in addition to the familiar weak-coupling phase. The existence of this new phase was exploited in a novel interpretation[2] of the multiple correlated and narrow-peak structures in electron and positron spectra[3] observed at GSI several years ago. According to this scenario, the e^+e^- peaks are due to the decay of a bound e^+e^- system formed in the new QED phase, which is induced by the strong and rapidly varying electromagnetic fields present in the neighborhood of the colliding heavy ions. While the experimental situation with regard to these anomalous e^+e^- events is unclear, especially after similar experiments at Argonne have yielded negative results[4], it is still of great interest to investigate whether background fields can be physically used to induce chiral symmetry breaking. Now the question is: what kind of background fields can potentially induce chiral symmetry breaking in gauge theories?[5] We recall that in a magnetic monopole field a gauge field breaks chiral symmetry[6] and that in the Nambu-Jona-Lasinio model a magnetic field drives the critical transition point towards weaker coupling[7]. Thus, magnetic fields are obvious candidates. We will take advantage of the strong-field techniques introduced by Schwinger and others to consider a constant magnetic field of arbitrary strength. In order to put our problem in as general a setting as possible, we will use the nonperturbative Schwinger-Dyson equation approach. We will then compare our result with that obtained recently by Gusynin, Miransky, and Shovkovy[8] whose

method is very different from ours. A comparison of the two approaches will sharpen our understanding of the underlying physics and the kind of approximations involved.

The motion of a massless fermion of charge e in an external electromagnetic field is described by the Green's function that satisfies the modified Dirac equation proposed by Schwinger:

$$\gamma \cdot \Pi(x) G_A(x, y) + \int d^4x' M(x, x') G_A(x', y) = \delta^{(4)}(x - y), \tag{1}$$

where $\Pi_\mu(x) = -i\partial_\mu - eA_\mu(x)$, and $M(x, x')$ is the mass operator M in the coordinate representation. For a constant magnetic field of strength H, we may take $A_2 = Hx_1$ to be the only nonzero component of A_μ. In the following we will use the method due to Ritus[9], which is based on the use of the eigenfunctions of the mass operator and the diagonalization of the latter. As shown by Ritus, M is diagonal in the representation of the eigenfunctions $E_p(x)$ of the operator $(\gamma \cdot \Pi)^2$:

$$-(\gamma \cdot \Pi)^2 E_p(x) = p^2 E_p(x). \tag{2}$$

The advantage of using this representation is obvious: M can now be put in terms of its eigenvalues, so the problems arising from its dependence on the operator Π can be avoided. In the chiral representation in which σ_3 and γ_5 are diagonal with eigenvalues $\sigma = \pm 1$ and $\chi = \pm 1$, respectively, the eigenfunctions $E_{p\sigma\chi}(x)$ take the form

$$E_{p\sigma\chi}(x) = N \mathrm{e}^{i(p_0 x^0 + p_2 x^2 + p_3 x^3)} D_n(\rho) \omega_{\sigma\chi} \equiv \tilde{E}_{p\sigma\chi} \omega_{\sigma\chi}, \tag{3}$$

where $D_n(\rho)$ are the parabolic cylinder functions[10] with indices

$$n = n(k, \sigma) \equiv k + \frac{eH\sigma}{2|eH|} - \frac{1}{2}, \qquad k = 0, 1, 2, ..., \tag{4}$$

and argument $\rho = \sqrt{2|eH|}(x_1 - \frac{p_2}{eH})$. Note that $n = 0,\ 1,\ 2,\ \ldots$. The normalization factor is $N = (4\pi|eH|)^{1/4}/\sqrt{n!}$; p stands for the set (p_0, p_2, p_3, k); and $\omega_{\sigma\chi}$ are the bispinors of σ_3 and γ_5.

Following Ritus, we form the orthonormal and complete[11] eigenfunction-matrices $E_p = \mathrm{diag}(\tilde{E}_{p11},\ \tilde{E}_{p-11},\ \tilde{E}_{p1-1},\ \tilde{E}_{p-1-1})$. They satisfy

$$\gamma \cdot \Pi\ E_p(x) = E_p(x)\ \gamma \cdot \bar{p} \tag{5}$$

and

$$M(x, x')E_p(x') = E_p(x)\delta^{(4)}(x - x')\tilde{\Sigma}_A(\bar{p}), \tag{6}$$

where $\tilde{\Sigma}_A(\bar{p})$ represents the eigenvalues of the mass operator, and $\bar{p}_0 = p_0,\ \bar{p}_1 = 0,\ \bar{p}_2 = -\mathrm{sgn}(eH)\sqrt{2|eH|k},\ \bar{p}_3 = p_3$. These properties of the $E_p(x)$ allow us to express the Green's function and the mass operator in the E_p-representation as ($\bar{E}_p \equiv \gamma^0 E_p^\dagger \gamma^0$)

$$G_A(x, y) = \sum\!\!\!\!\!\!\int \frac{d^4p}{(2\pi)^4} E_p(x) \frac{1}{\gamma \cdot \bar{p} + \tilde{\Sigma}_A(\bar{p})} \bar{E}_p(y), \quad \sum\!\!\!\!\!\!\int d^4p \equiv \sum_k \int dp_0 dp_2 dp_3, \tag{7}$$

and

$$M(p, p') = \int d^4x d^4x' \bar{E}_p(x) M(x, x') E_{p'}(x') = \tilde{\Sigma}_A(\bar{p})(2\pi)^4 \hat{\delta}^{(4)}(p - p'), \tag{8}$$

respectively, where $\hat{\delta}^{(4)}(p - p') \equiv \delta_{kk'}\delta(p_0 - p_0')\delta(p_2 - p_2')\delta(p_3 - p_3')$.

We work in the ladder approximation in which

$$M(x, x') = ie^2\gamma^\mu G_A(x, x')\gamma^\nu D_{\mu\nu}(x - x'), \tag{9}$$

where $D_{\mu\nu}(x - x')$ is the bare photon propagator,

$$D_{\mu\nu}(x - x') = \int \frac{d^4q}{(2\pi)^4} \frac{\mathrm{e}^{iq\cdot(x-x')}}{q^2 - i\epsilon} \left(g_{\mu\nu} - (1 - \xi)\frac{q_\mu q_\nu}{q^2} \right). \tag{10}$$

The Schwinger-Dyson (SD) equation then takes the form

$$\tilde{\Sigma}_A(\bar{p})(2\pi)^4\hat{\delta}(p-p') = ie^2\int d^4xd^4x'\sum\!\!\!\!\!\!\int \frac{d^4p"}{(2\pi)^4}\bar{E}_p(x)\gamma^\mu E_{p"}(x) \times\frac{1}{\gamma\cdot\bar{p"}+\tilde{\Sigma}_A(\bar{p"})}\bar{E}_{p"}(x')\gamma^\nu E_{p'}(x')D_{\mu\nu}(x-x'). \quad (11)$$

After integrations over $x,\ x',\ p"_0,\ p"_2$, and $p"_3$, the SD equation is simplified to read ($r\equiv\sqrt{(q_1^2+q_2^2)/(2|eH|)},\ \varphi\equiv\tan^{-1}(-q_2/q_1)$)

$$\begin{aligned}\tilde{\Sigma}_A(\bar{p})\delta_{kk'} &= ie^2\sum_{k"}\int\frac{d^4q}{(2\pi)^4}\frac{1}{\sqrt{n!n'!n"!\tilde{n}"!}}\mathrm{e}^{-r^2}\mathrm{e}^{i\ \mathrm{sgn}(eH)(n'-n+n"-\tilde{n}")\varphi}\\ &\times\frac{1}{q^2}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right)\gamma^0\Delta\gamma^0\gamma^\mu\Delta"\\ &\times\frac{1}{\gamma\cdot\bar{p"}+\tilde{\Sigma}_A(\bar{p"})}\gamma^0\tilde{\Delta}"\gamma^0\gamma^\nu\Delta' J_{nn"}(r)J_{\tilde{n}"n'}(r), \quad (12)\end{aligned}$$

where summing over $\sigma,\ \sigma',\ \sigma"$, and $\tilde{\sigma}"$ on the right hand side is understood, and

$$J_{nn'}(r)\equiv\sum_{m=0}^{\min(n,n')}\frac{n!n'!}{m!(n-m)!(n'-m)!}[i\ \mathrm{sgn}(eH)r]^{n+n'-2m}. \quad (13)$$

We have also used the following notations[12] in Eq.(12): $\bar{p"}_0=p_0-q_0$, $\bar{p"}_1=0$, $\bar{p"}_2=-\ \mathrm{sgn}(eH)\sqrt{2|eH|k"}$, $\bar{p"}_3=p_3-q_3$, $\Delta=\Delta(\sigma)=\mathrm{diag}(\delta_{\sigma1},\delta_{\sigma-1},\delta_{\sigma1},\delta_{\sigma-1})$, $\Delta'=\Delta(\sigma')$, ... etc., $n'=n(k',\sigma')$, $n"=n(k",\sigma")$, and $\tilde{n}"=n(k",\tilde{\sigma}")$.

Eq.(12) may be solved by following the standard procedure[13] of writing $\tilde{\Sigma}_A(\bar{p})=\beta\gamma\cdot\bar{p}+\Sigma_A(\bar{p})$, where $\Sigma_A(\bar{p})$ corresponds to the dynamically generated fermion mass. We will assume that $\Sigma_A(\bar{p})$ is proportional to the unit matrix (it will be seen later from the solution that this is a self-consistent assumption). Eq.(12) then leads to two coupled equations for β and Σ_A:

$$\begin{pmatrix}\Sigma_A(\bar{p})\\ \beta\gamma\cdot\bar{p}\end{pmatrix}\delta_{kk'} = ie^2\sum_{k"}\int\frac{d^4q}{(2\pi)^4}\frac{1}{\sqrt{n!n'!n"!\tilde{n}"!}}\mathrm{e}^{-r^2}\mathrm{e}^{i\ \mathrm{sgn}(eH)(n'-n+n"-\tilde{n}")\varphi}$$

$$\times J_{nn"}(r) J_{\tilde{n}"n'}(r) \frac{1}{(1+\beta)^2 \bar{p"}^2 + \Sigma_A^2(\bar{p"})}$$
$$\times \frac{1}{q^2} \begin{pmatrix} \Sigma_A(\bar{p"})(G_1 - \frac{1-\xi}{q^2} Q_1) \\ -(1+\beta)(G_2 - \frac{1-\xi}{q^2} Q_2) \end{pmatrix} \tag{14}$$

where $G_1 = \Delta \gamma^\mu \Delta" \tilde{\Delta}" \gamma_\mu \Delta' = -2(\delta_{\sigma"1}\delta_{\tilde{\sigma}"1} + \delta_{\sigma"-1}\delta_{\tilde{\sigma}"-1}) \mathrm{diag}(\delta_{\sigma 1}\delta_{\sigma' 1}, \delta_{\sigma -1}\delta_{\sigma' -1}, \delta_{\sigma 1}\delta_{\sigma' 1}, \delta_{\sigma -1}\delta_{\sigma' -1})$, $Q_1 = \Delta(\gamma \cdot q)\Delta" \tilde{\Delta}" (\gamma \cdot q)\Delta'$, $G_2 = \Delta \gamma^\mu \Delta" (\gamma \cdot \bar{p"}) \tilde{\Delta}" \gamma_\mu \Delta'$, and $Q_2 = \Delta(\gamma \cdot q)\Delta" (\gamma \cdot \bar{p"}) \tilde{\Delta}" (\gamma \cdot q)\Delta'$.

We seek solutions with $\beta = 0$. We will show later that such a solution is consistent only with the Feynman gauge ($\xi = 1$). In this case the two SD equations decouple and only G_1 is relevant for determining the dynamical fermion mass. The spin structure of G_1 implies that necessarily $\sigma" = \tilde{\sigma}"$, which, in turn, implies that necessarily $n" = \tilde{n}"$. It is convenient to make a change of integration variables from (q_1, q_2) to the "polar coordinates" (r, φ). The integration over φ yields

$$\int_0^{2\pi} d\varphi \mathrm{e}^{i \ \mathrm{sgn}(eH)(n'-n)\varphi} = 2\pi \delta_{nn'} \tag{15}$$

We note that the spin structure of G_1 also implies that $\sigma = \sigma'$, which, together with the $\delta_{nn'}$ from Eq.(15), matches the $\delta_{kk'}$ on the left hand side of Eq.(14).

Due to the factor e^{-r^2} in the integrand in Eq.(14), contributions from large values of r are suppressed. Let us therefore, as an approximation (we will find out later what physical condition validates this approximation), keep only the smallest power of r in $J_{nn"}(r)$, i.e.,

$$J_{nn"}(r) \to \frac{[\max(n, n")]!}{|n - n"|!} (i \ \mathrm{sgn}(eH) r)^{|n-n"|}. \tag{16}$$

Since the leading contributions come from the term corresponding to $n" = n$, we need only keep the term given by $k" = n+\frac{1}{2}-\frac{\sigma"}{2}\mathrm{sgn}(eH)$ in the summation over $k"$. As a result, we can replace $J_{nn"}$ by $n!$. The SD equation (Eq.(14)), thereby vastly simplified, becomes

$$\Sigma_A(\bar{p}) \simeq \frac{ie^2}{(2\pi)^3}|eH| \int dq_0 dq_3 \int_0^\infty dr^2 \mathrm{e}^{-r^2} \frac{G_1}{q^2} \frac{\Sigma_A(\bar{p"})}{\bar{p"}^2 + \Sigma_A(\bar{p"})} \tag{17}$$

where $q^2 = -q_0^2 + q_3^2 + 2|eH|r^2$ and $\bar{p"}^2 = -(p_0 - q_0)^2 + (p_3 - q_3)^2 + 2|eH|k"$.

Let us make a Wick rotation to Euclidean space: $p_0 \to ip_4$, $q_0 \to iq_4$. Consider the case with $p = 0$, i.e., $p_0 = p_3 = k = 0$. Notice that $k = 0$ means that, for positive (negative) eH, $\sigma = 1(-1)$ and $n = 0$, the last of which implies that $k" = 0$ and $\sigma" = 1(-1)$ for the respective sign of eH. We also note that for either sign of eH, the matrix G_1 can be effectively replaced by $-2 \times \mathbf{1}$. We will assume that the dominant contributions to the integral in Eq.(17) come from the infrared region of small q_3 and q_4 (this assumption will be seen to be self-consistent). Thus, it is reasonable to replace $\Sigma_A(\bar{p"})$ in the integrand by $\Sigma_A(0) = m \times \mathbf{1}$. Eq.(17) then becomes

$$m \simeq \frac{\alpha}{\pi^2} \int dq_3 dq_4 \int_0^\infty dr^2 \mathrm{e}^{-r^2} \frac{1}{2r^2 + L^2(q_3^2 + q_4^2)} \frac{m}{m^2 + (q_3^2 + q_4^2)} \tag{18}$$

where $\alpha = e^2/4\pi$ is the fine structure constant and $L = 1/\sqrt{|eH|}$ is the magnetic length. The integrations over q_3 and q_4 give

$$1 \simeq \frac{\alpha}{\pi} \int_0^\infty dr^2 \frac{\mathrm{e}^{-r^2} \ln(2r^2/m^2L^2)}{2r^2 - m^2L^2} \tag{19}$$

which yields the nonzero dynamical mass as

$$m \simeq a \sqrt{|eH|}\, \mathrm{e}^{-b\sqrt{\frac{\pi}{\alpha} - c}}, \tag{20}$$

where a, b, and c are constants of order 1.

Eq.(20) clearly demonstrates the nonperturbative nature of the result. It also shows that our approximations break down when $\alpha > O(1)$. As a further check on the consistency of our assumptions, we note that, according to Eq.(18), the dominant contributions to the integrals come from the region $2r^2 \sim m^2L^2 \sim L^2(q_3^2 + q_4^2)$. Our earlier assumption that effectively $r \ll 1$ is now translated to the physical assumption that $mL \ll 1$, which requires that $\alpha \ll O(1)$; in other words, the dynamical chiral symmetry breaking solution we have found applies to the weak-coupling regime of QED! Now it is also evident that indeed the infrared region of q_3 and q_4 gives the dominant contributions to the integrals.

It remains for us to show that $\beta = 0$ solves Eq. (14) only if $\xi = 1$. The main point to note is that, consistent with the $k" = 0$ approximation made above, we can approximate the $\gamma \cdot \bar{p}"$ in G_2 by $-(\gamma^0 q_0 + \gamma^3 q_3)$ (recall also that we are considering the case of $p = 0$). But then the piece of the integrand involving G_2 is odd in q_0 as well as in q_3, and hence vanishes upon integration. It follows that the solution $\beta = 0$ requires $\xi = 1$, i.e., the Feynman gauge. As a result, the Q_1-piece on the right hand side of Eq. (14) does not contribute. (That is fortunate because Q_1 is actually not proportional to the unit matrix; its presence in the SD equation would have spoiled the assumption that Σ_A is the dynamical mass multiplied by the unit matrix.)

In summary, we have found a solution to the Schwinger-Dyson equation in the presence of an arbitrarily strong constant magnetic field, which indicates that, even at weak gauge coupling, an external magnetic field can trigger the dynamical breaking of chiral symmetry in QED, with the dynamical

mass of the fermion given by Eq.(20). Our general conclusion agrees with a recent finding by Gusynin *et al.*[8], whose approach is very different from ours. It would be interesting to examine if there are additional solutions of chiral symmetry breaking due to an external magnetic field. A parallel calculation for the case of a constant background electric field[14] or other background field configurations may also shed light on the dynamics of chiral symmetry breaking in gauge theories. The formalism proposed here will be most suitable for these studies.

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[11] The orthonormality property reads $\int d^4x \bar{E}_{p'}(x) E_p(x) = (2\pi)^4 \hat{\delta}^{(4)}(p-p')$ and the completeness property is as contained in $\sum\!\!\!\!\int d^4p E_p(x) \bar{E}_p(y) = (2\pi)^4 \delta^{(4)}(x-y)$.

[12] In the absence of an electric field, we have $\tilde{E}_{p11} = \tilde{E}_{p1-1}$ and $\tilde{E}_{p-11} = \tilde{E}_{p-1-1}$ so that references to χ are irrelevant and can be dropped.

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